A note on monomial ideals[1]

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Abstract We show that the number of elements generating a squarefree monomial ideal up to radical can always be bounded above in terms of the number of its minimal monomial generators and the maximal height of its minimal primes.

Introduction

Let R be a commutative Noetherian ring with non-zero identity. If I is an ideal of R, we say that the elements $a_1, \ldots, a_s \in R$ *generate* I *up to radical* if $\mathrm{Rad}\,(a_1, \ldots, a_s) = \mathrm{Rad}\,I$. The minimum number s with this property is called the *arithmetical rank* of I. If $\mu(I)$ is the minimum number of generators for I, then obviously

$$\mathrm{ara}\,I \leq \mu(I). \tag{1}$$

This inequality yields a trivial upper bound for the arithmetical rank of I. Moreover, it is well-known that

$$h(I) \leq \mathrm{ara}\,I, \tag{2}$$

where $h(I)$ is the height of I. If equality holds in (1) and in (2), I is called a *complete intersection*. The following result by Schmitt and Vogel [12], p. 249, shows that (1) can be improved if special combinatorial conditions on the generating set are fulfilled.

Lemma 1 *Let P be a finite subset of elements of R. Let $P_0, \ldots, P_r$ be subsets of P such that*

(i) $\bigcup_{i=0}^{r} P_i = P$;

(ii) P_0 *has exactly one element;*

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(iii) if p and p' are different elements of P_i $(0 < i \le r)$ there is an integer i' with $0 \le i' < i$ and an element in $P_{i'}$ which divides pp'.

We set $q_i = \sum_{p \in P_i} p^{e(p)}$, where $e(p) \ge 1$ are arbitrary integers. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\mathrm{Rad}\,(P) = \mathrm{Rad}\,(q_0, \ldots, q_r).$$

We will apply Lemma 1 in the polynomial ring $R = K[x_1, \ldots, x_n]$, where K is a field, in order to determine upper bounds for the arithmetical rank of ideals generated by monomials, i.e., by products of indeterminates. Since ideals with the same radical have the same arithmetical rank, without loss of generality we can restrict our investigation to radical monomial ideals. These are the so-called *squarefree* monomial ideals, i.e., the ideals that are generated by products of pairwise distinct indeterminates.

The aim of this paper is to replace inequality (1) by a more precise one in the case where I is a squarefree monomial ideal. We will derive a closed expression, depending only on $\mu(I)$ and the heights of the minimal primes of I, which can be put on the right-hand of inequality (1) in order to obtain a better general upper bound. This new upper bound is still sharp when I is a complete intersection. Moreover it improves the general upper bound conjectured by Lyubeznik in [9] when I has a small number of generators.

Recall that every monomial ideal I has a unique monomial generating set of cardinality $\mu(I)$: it is the one formed by the monomials in the ideal that are minimal with respect to the divisibility relation.

1 A general upper bound

Let I be a squarefree monomial ideal of R, and let M be the set of $\mu(I)$ monomial generators for I. For all $i = 1, \ldots, n$ let

$$M_i = \{f \in M \mid x_i \text{ divides } f\}. \tag{3}$$

Let $I_1, \ldots, I_r$ be the minimal primes of I, so that $I = \cap_{j=1}^r I_j$. For all $j = 1, \ldots, r$ set

$$\nu_j = \max\{|M_i| \mid x_i \in I_j\}, \tag{4}$$

and let $x_{i_j} \in I_j$ be such that $\nu_j = |M_{i_j}|$. Here the bars indicate the cardinality. Finally, let

$$\nu(I) = \min\{\nu_j \mid j = 1, \ldots, r\}. \tag{5}$$

From definitions (3), (4) and (5) it follows that $\nu(I) \le |M| = \mu(I)$.

Proposition 1 $ara\, I \leq \mu(I) - \nu(I) + 1$.

Proof .-For the sake of simplicity we put $\mu = \mu(I)$ and $\nu = \nu(I)$. We are going to explicitly construct a set of at most $\mu - \nu + 1$ polynomials which generate I up to radical. For all $k = 1, \ldots, \mu$ let

$$L_k = \{\prod_{i=1}^{k} f_i \mid f_1, \ldots, f_k \text{ are pairwise distinct elements of } M\}.$$

Further, let

$$p_0 = \prod_{j=1}^{r} x_{i_j} \in \cap_{j=1}^{r} I_j = I.$$

Then put

$$P_0 = \{p_0\}, \tag{6}$$

and, for all $i = 1, \ldots, \mu - \nu$,

$$P_i = L_{\mu - \nu + 1 - i}.$$

We show that, for all $i = 1, \ldots, \mu - \nu$,

the product of any two distinct elements of P_i is divisible by some element of P_{i-1}. (7)

We first show it for $i > 1$. Set $k = \mu - \nu + 1 - i$ and let p, p' be distinct elements of $P_i = L_k$. Then there are two different k-subsets $\{f_1, \ldots, f_k\}$ and $\{f_1', \ldots, f_k'\}$ of M such that

$$p = \prod_{i=1}^{k} f_i \qquad \text{and} \qquad p' = \prod_{i=1}^{k} f_i'.$$

Up to a change of indices we may assume that $f_1' \notin \{f_1, \ldots, f_k\}$. Then pp' is divisible by

$$\prod_{i=1}^{k} f_i \cdot f_1' \in L_{k+1} = P_{i-1}.$$

Now consider two distinct elements p, p' of $P_1 = L_{\mu - \nu}$. By the above argument, pp' is divisible by a product π of at least $\mu - \nu + 1$ distinct elements of M. By definition, for all $j = 1, \ldots, r$, exactly ν_j elements of M are divisible by x_{i_j}, in particular, by definition (5), at least ν elements of M are divisible by x_{i_j}, or, equivalently, at most $\mu - \nu$ are not divisible by x_{i_j}. It follows that the product π necessarily involves, as a factor, an element of M which is divisible by x_{i_j}. Hence pp' is divisible by l_0.
This proves condition (7) and implies that $P_0, \ldots, P_{\mu - \nu}$ fulfill the assumption (iii) of Lemma 1 with $i' = i - 1$. Moreover, we have that $P_{\mu - \nu} = L_1 = M$, and $P_i \subset (M)$ for

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all $i = 0, \ldots, \mu - \nu$, so that $\left(\cup_{i=0}^{\mu-\nu} P_i \right) = (M) = I$. Thus assumption (i) of Lemma 1 is satisfied by $P = M$. Finally, assumption (ii) is trivially true by (6). Therefore, if we set

$$q_i = \sum_{p \in P_i} p \qquad \text{for } i = 0, \ldots, \mu - \nu,$$

by Lemma 1 we have that

$$I = (M) = \text{Rad}\,(q_0, \ldots, q_{\mu-\nu}),$$

which implies the claim.

Remark 1 The construction of the sets P_i given in the proof of Lemma 1 does not give rise to an efficient algorithm, since it requires the computation of $2^{\mu(I)} - 1$ polynomial products. The output can be simplified if one reduces each set P_i (with $i \geq 1$) to the set of its elements which are minimal with respect to the divisibility relation. This will not affect condition (7), and will possibly produce a smaller number of polynomials q_i. Sets P_i fulfilling (7) and assumptions (i) and (ii) of Lemma 1 can also be constructed by the following method, which is taken from [1], and is more convenient from a computational point of view. We first put $\Gamma_1 = M$, and for all indices $i \geq 2$, we recursively define Γ_i as the set of all elements of

$$\{\text{lcm}\,(f, g) \mid f, g \in \Gamma_{i-1}, f \neq g\}$$

which are are minimal with respect to the divisibility relation. The procedure stops at step N as soon as there is some $p_0 \in I$ which divides all elements of Γ_N. At this point we set $P_0 = \{p_0\}$ and $P_i = \Gamma_{N-i}$ for all $i = 1, \ldots, N - 1$. Then the above condition (7) is trivially true.

The different approaches discussed here are not equivalent: one can find examples where they yield distinct numbers of polynomials generating I up to radical.

Next we show that number $\mu(I)$ can be compared with $\nu(I)$. Let

$$\tau(I) = \max\{h(I_j) \mid j = 1, \ldots, r\}.$$

Then we have

Proposition 2 $\mu(I) \leq \nu(I)\tau(I).$

Proof .-We adopt the simplified notation used in the preceding proof and we also set $\tau = \tau(I)$. In view of (5) we have to prove that $\mu \leq \nu_j \tau$ for all $j = 1, \ldots, r$. We show it for $j = 1$. Up to renaming the indeterminates we may assume that $I_1 = (x_1, \ldots, x_s)$ for some $s \leq \tau$. Since $M \subset I_1$, by (3) we have $M = \cup_{i=1}^{s} M_i$, whence

$$\mu = |M| \leq \sum_{i=1}^{s} |M_i| \leq s\nu_1 \leq \tau\nu_1,$$

where the second inequality follows from (4) for $j = 1$. This completes the proof.

The next result is an immediate consequence of Propositions 1 and 2.

Corollary 1 *Let J be an ideal of R such that $I = \mathrm{Rad}\,(J)$ is a (squarefree) monomial ideal. Then*

$$\mathrm{ara}\,J \leq \mu(I) - \frac{\mu(I)}{\tau(I)} + 1. \tag{8}$$

Remark 2 The new upper bound given in Corollary 1 is, like the one in (1), sharp if $I = J$ and I is a complete intersection: in that case $\mu(I) = h(I)$ and it is well-known that I is pure-dimensional, i.e., all its minimal primes have the same height (see, e.g., [2], Corollary 5.1.5), so that $h(I) = \tau(I)$.

Remark 3 For a squarefree monomial ideal I, the number $\mu(I)$ can be bounded above in terms of the number n of indeterminates of R and/or of the heights of the minimal primes of I. Some of these formulas can be found in [5] or in [6]. If we replace them in (8) we obtain similar upper bounds for $\mathrm{ara}\,I$. For example, Lemma (4) (a) in [5] yields

$$\mathrm{ara}\,I \leq \left(1 - \frac{1}{\tau(I)}\right)\binom{n}{[\frac{n}{2}]} + 1,$$

where the square brackets denote the integer part, and, according to Satz (5) (a) in [5], we have

$$\mathrm{ara}\,I \leq \left(1 - \frac{1}{\tau(I)}\right)\binom{n}{h(I) - 1} + 1,$$

whenever $h(I) \geq \frac{n+1}{2}$.

2 On a conjecture by Lyubeznik

It was conjectured by Lyubeznik in [9] that for every pure-dimensional ideal J of R of height t,

$$\operatorname{ara} J \le n - \left\lceil \frac{n-1}{t} \right\rceil, \tag{9}$$

so that, in particular, for every pure-dimensional monomial ideal I,

$$\operatorname{ara} I \le n - \left\lceil \frac{n-1}{\tau(I)} \right\rceil. \tag{10}$$

Inequality (9) has been proven by the same author in [10] for all ideals J in the localized ring $R_{(x_1,\dots,x_n)}$ (if K is infinite), whereas (10) was established in [7], Theorem 6, for monomial ideals I with $\tau(I) = 2, 3$ (see also [1], Section 2, for a different proof in the case $\tau(I) = 2$). Other special cases were examined in [11] and [12]. In [1] it is conjectured that (10) holds for every squarefree monomial ideal I. The results in the previous section allow us to show, and even improve, inequality (10) for a new class of monomial ideals. Corollary 1 implies the following:

Corollary 2 *Let J be an ideal of R such that $I = \operatorname{Rad}(J)$ is a monomial ideal generated by at most $n - 1$ (squarefree) monomials. Then*

$$\operatorname{ara} J \le n - \frac{n-1}{\tau(I)}.$$

Remark 4 In the class of squarefree monomial ideals I such that $\mu(I) \le n - 1$ the upper bounds given in Proposition 1 and in Corollary 1 are in general strictly better than the one in (10), as the next example shows.

Example 1 Suppose that char $K = 0$ and in $R = K[x_1, \dots, x_9]$ consider the ideal

$$I = (x_1 x_2,\ x_1 x_3,\ x_2 x_4,\ x_4 x_5,\ x_4 x_6,\ x_2 x_7,\ x_6 x_8,\ x_6 x_9),$$

with $\mu(I) = 8$. Its minimal primes can be quickly computed by CoCoA [3]:

$$I_1 = (x_1, x_2, x_4, x_6), \qquad I_2 = (x_2, x_3, x_4, x_6), \qquad I_3 = (x_1, x_2, x_5, x_6),$$

$$I_4 = (x_2, x_3, x_5, x_6), \qquad I_5 = (x_1, x_4, x_6, x_7), \qquad I_6 = (x_1, x_2, x_4, x_8, x_9),$$

$$I_7 = (x_2, x_3, x_4, x_8, x_9), \qquad I_8 = (x_1, x_4, x_7, x_8, x_9).$$

Thus we have $\tau(I) = 5$, so that $n - [\frac{n-1}{\tau(I)}] = 9 - [\frac{8}{5}] = 8$. Hence (10) yields the trivial inequality (1). From Corollary 1 we derive that $\mathrm{ara}\, I \le 8 - \frac{8}{5} + 1 = 7.4$, i.e., $\mathrm{ara}\, I \le 7$. Moreover,

$$|M_3| = |M_5| = |M_7| = |M_8| = |M_9| = 1, \qquad |M_1| = 2,$$
$$|M_2| = |M_4| = |M_6| = 3,$$

so that, by definition (4), $\nu_j = 3$ for all $j = 1, \ldots, 8$. Therefore, by definition (5), $\nu(I) = 3$. Hence Proposition 1 gives us $\mathrm{ara}\, I \le 8 - 3 + 1 = 6$. The method from [1] that we mentioned in Remark 1 also yields $\mathrm{ara}\, I \le 6$. In fact $\mathrm{ara}\, I = 5$: on the one hand, by virtue of Lemma 1, I is generated up to radical by the 5 polynomials

$$x_1x_2, \ x_1x_3 + x_2x_4, \ x_2x_7 + x_4x_6, \ x_4x_5 + x_6x_8, \ x_6x_9;$$

on the other hand, according to [8], Theorem 1, and [4], Example 2, p. 414, we have $\mathrm{pd}\,(R/I) \le \mathrm{ara}\, I$, where pd denotes the projective dimension, and CoCoA tells us that here $\mathrm{pd}\,(R/I) = 5$.

In the previous example, the best upper bound for the arithmetical rank is the one derived from Proposition 1. Sometimes this proposition yields the exact value of the arithmetical rank. An interesting example of this kind can be obtained by a slight modification of the ideal in Example 1.

Example 2 Suppose that char $K = 0$ and in $R = K[x_1, \ldots, x_9]$ consider the ideal

$$I = (x_1x_2, \ x_1x_3, \ x_2x_4, \ x_4x_5, \ x_4x_6, \ x_6x_7, \ x_6x_8, \ x_6x_9),$$

with $\mu(I) = 8$. The minimal primes of I are

$$I_1 = (x_1, x_4, x_6), \qquad I_2 = (x_2, x_3, x_4, x_6), \qquad I_3 = (x_1, x_2, x_5, x_6),$$

$$I_4 = (x_2, x_3, x_5, x_6), \qquad I_5 = (x_1, x_4, x_7, x_8, x_9), \qquad I_6 = (x_2, x_3, x_4, x_7, x_8, x_9).$$

Now $\tau(I) = 6$, so that $n - [\frac{n-1}{\tau(I)}] = 9 - [\frac{8}{6}] = 8$. As in the previous example, (10) yields a trivial upper bound. From Corollary 1 we deduce that $\mathrm{ara}\, I \le 8 - \frac{8}{6} + 1 = 7.\bar{6}$, i.e., $\mathrm{ara}\, I \le 7$. Since $\nu(I) = 3$, by Proposition 1 we again have $\mathrm{ara}\, I \le 6$. The same result is obtained if we use the method from [1], and also if we apply Lemma 1 directly to the minimal monomial generators of I: I is generated up to radical by the 6 polynomials

$$x_2x_4, \ x_1x_2 + x_4x_6, \ x_4x_5 + x_6x_7, \ x_1x_3, \ x_6x_8, \ x_6x_9.$$

This time $\mathrm{pd}\,(R/I) = 6$, so that $\mathrm{ara}\, I = 6$.

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